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Subspace Structures of Holomorphic Representations

ABSTRACT. *A set of basic covariant differential operators between holomorphic (»positive energy«) representations on hermitian symmetric spaces is described. The set is in a bijective correspondence with a collection of particularly fundamental homomorphisms between highest weight modules. A method, by which one may approach the general situation from this, is presented. A symmetry principles is also introduced.*

Secondly, two different natural ways of producing irreducible mixtures of unitarizable highest weight modules are exemplified. One is by means of restriction to a maximal parabolic subgroup, the other is through the imbedding of one hermitian symmetric space into another.

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Introduction

In the perhaps vain hope that the following may be read, at least in part, by somebody from outside the mathematical community, we begin by making a few general remarks about mathematics, and after that, about the area of mathematics to which the following belongs. Only then, and with somewhat less pathos, do we become specific and turn to the content of the present paper.

Let us then assert, with a slight reformulation of a definition in the fascinating book »The Mathematical Experience« by Davis and Hersh ([D&H]) that mathematics is the science of spaces and numbers. In rather simple terms, and in analogy with everyday life, one defines objects, and lays down rules according to which one may manipulate with, or operate upon, these. (Or, the analogy is to a property of an object which was defined in analogy with everyday life. Etc.). Through logical deduction one then tries to reveal deep and non-obvious properties of these creations. One develops tools for the investigation, one invents (or, as many are inclined to say, discovers) new models, and one classifies, calculates on, solves problems on, dissects, welds together, and even sometimes performs experiments on, these structures. Led by an intuition founded on logic, and in the firm belief that abstraction goes in the direction of simplicity, clarity, and verity, theories of great intricacy and beauty are created. Included among the objects of mathematical interest have always been those which at a given time have been considered to correspond to »reality«, but mathematics is much richer. At the same time it should be mentioned that to many mathematicians, their objects are real and have as much right to be called such, as more down to earth fundamentals.

The area of mathematics to which the following belongs is the representation theory of semi-simple Lie groups.

A typical way in which a Lie group emerges is if one has a set M carrying, or equipped with, a certain structure (e.g. a differential equation on $\mathbb{R}^n$). The group of maps of M onto M that preserve this structure is then often a Lie group. Closely related to this is the occurrence of symmetry groups in physics. The Poincaré group is, for instance, the group of causality (and scale) preserving transformations of Minkowski space. Another interesting example is the conformal group SU(2,2). In fact, the investigations of Segal into the notions of time and causality that led him to propose the conformal group as a possible fundamental symmetry group ([S], see also [S,J,Ø,P,&S] and references cited therein) was what motivated us to persue the kind of representation theory presented below.

Specifically, we are concerned with representations living in spaces of vector valued holomorphic functions on a hermitian symmetric space $\mathscr{D}$ of the non-compact type. The property of holomorphy is closely related to the physical concept of positivity of the energy. For simplicity assume that $\mathscr{D}$ is a tube domain; $\mathscr{D} = \mathbb{R}^n + iC^+$, where C^+ is an open proper convex cone in $\mathbb{R}^n$. Then $\mathbb{R}^n$ is the Shilov boundary of $\mathscr{D}$ and in the regular (generic) case the spaces of holomorphic functions that carry the unitary representations of the group G of holomorphic transformations of $\mathscr{D}$, are Fourier-Laplace transforms of spaces of functions living on C^+.

The current article deals with the subspace structure of such representations, also outside the realm of unitarity. Chapter 1 is mainly concerned with invariant subspaces defined by covariant differential operators. Let us take time here to stress that even though the formulation is infinitesimal, one can always quite easily integrate to an appropriate covering group of G.

We describe in Section 1.2 what we call the basic covariant differential operators on $\mathscr{D}$, namely those which originate or terminate in scalar modules. In Section 1.3 we indicate how one can approach the general situation by means of these results, and finally Section 1.4 is concerned with the conformal group, where the analysis can be brought to a full conclusion. At the same time a principle, which we believe will be fundamental for the further investigations, is introduced.

Chapter 2 deals with irreducible mixtures of unitarizable modules. Through some simple examples, two different situations are presented where one, in a natural way, encounters such a phenomenon. The first circumstance is with representations which, when restricted to a maximal parabolic subgroup, decompose into a finite sum of irreducibles, and the other occurs when a hermitian symmetric space is embedded compatibly into a bigger one. It is remarkable how rich the structures are that result from such simple phenomena.

1. Covariant differential operators

1.1. Fundamentals

Let $\mathfrak{g}$ denote the Lie algebra of the group of holomorphic transformations of an irreducible hermitian symmetric space $\mathscr{D}$. It is well-known that $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{R}$ and that there are compact Cartan subalgebras. Specifically, let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Then $\mathfrak{k}$ has a one-dimensional center η. Let h_0 denote one of the two elements of η whose eigenvalues on $\mathfrak{p}^{\mathbb{C}}$ are $\pm$ i, and let $\mathfrak{p}^+$ and $\mathfrak{p}^-$ denote the $+i$ and $-i$ eigenspace, respectively, for this fixed element. Let $\mathfrak{k}_1 = [\mathfrak{k},\mathfrak{k}]$ denote the semi-simple part of $\mathfrak{k}$ and let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{k}$. Then $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathbb{R}\cdot h_0$, $\mathfrak{h} = (\mathfrak{h}\cap\mathfrak{k}_1) \oplus \mathbb{R}\cdot h_0$, $(\mathfrak{h}\cap\mathfrak{k}_1)^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{k}_1^{\mathbb{C}}$, and $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

The sets of compact and non-compact roots of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}^{\mathbb{C}}$ are denoted Δ_c and Δ_n, respectively; $\Delta = \Delta_c \cup \Delta_n$. We choose an ordering of Δ such that $\mathfrak{p}^+$ corresponds to Δ_n^+. Throughout β denotes the unique simple root in Δ_n^+ and ϱ denotes one half of the sum of the positive roots. For $\gamma \in \Delta$ let H_γ denote the unique element of $i\mathfrak{h} \cap [(\mathfrak{g}^{\mathbb{C}})^\gamma, (\mathfrak{g}^{\mathbb{C}})^{-\gamma}]$ for which $\gamma(H_\gamma) = 2$. Finally, following [R&V] we let γ_r denote the highest root. Then $\gamma_r \in \Delta_n^+$ and $H_{\gamma_r} \notin [\mathfrak{h}\cap\mathfrak{k}_1]^{\mathbb{C}}$.

If Λ_0 is a dominant integral weight of $\mathfrak{k}_1$ and if $\lambda \in \mathbb{R}$ we denote by $\Lambda = (\Lambda_0,\lambda)$ the linear functional on $\mathfrak{h}^{\mathbb{C}}$ given by

$$\Lambda|_{(\mathfrak{h}\cap\mathfrak{k}_1)^{\mathbb{C}}} = \Lambda_0,\ \Lambda(H_{\gamma_r}) = \lambda. \tag{1.1.1}$$

Such a Λ determines an irreducible finite-dimensional $\mathcal{U}(\mathfrak{k}^{\mathbb{C}})$-module which we, for convenience, denote by V_τ. Here $\tau = \tau_\Lambda$ denotes the representation corresponding to Λ of the connected simply connected Lie group $\widetilde{K}$ with Lie algebra $\mathfrak{k}$. Further, let

$$M(V_\tau) = \mathcal{U}(\mathfrak{g}^{\mathbb{C}}) \underset{\mathcal{U}(\mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{+})}{\otimes} V_\tau \tag{1.1.2}$$

denote the generalized Verma module of highest weight Λ, and let M_Λ denote the Verma module of which $M(V_\tau)$ is a quotient.

In what follows, we choose to represent our Hermitian symmetric space $\mathcal{D}$ as a bounded domain in $\mathfrak{p}^-$. Consider an (irreducible) finite-dimensional $\mathcal{U}(\mathfrak{k}^{\mathbb{C}})$-module V_τ. Through the process of holomorphic induction, the space $\mathcal{P}(V_\tau)$ of V_τ-valued polynomials on $\mathfrak{p}^-$ becomes a $\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$-module consisting of $\mathfrak{k}$- (or $\widetilde{K}$-) finite vectors. We maintain the notation $\mathcal{P}(V_\tau)$ for this module and let dU_τ denote the corresponding representation of $\mathfrak{g}^{\mathbb{C}}$. Explicitly, let

$$(\delta(z_0)f)\ (z) = \frac{d}{dt}\ \Big|_{t=0} f(z+tz_0)$$

for $z_0, z \in \mathfrak{p}^-$, and $f \in C^\infty(\mathfrak{p}^-)$. Then, for $p \in \mathcal{P}(V_\tau)$ we have ([J&V; II]):

$$\begin{aligned}
(dU_\tau(x)f)\ (z) &= -\ (\delta(x)f)\ (z) && \text{for } x \in \mathfrak{p}^-, \quad (1.1.3)\\
(dU_\tau(x)f)\ (z) &= d\tau(x)f(z) - (\delta([x,z])f)\ (z) && \text{for } x \in \mathfrak{k}^{\mathbb{C}}, \text{ and}\\
(dU_\tau(x)f)\ (z) &= d\tau([x,z])f(z) - \tfrac{1}{2}(\delta([[x,z],z])f)\ (z) && \text{for } x \in \mathfrak{p}^{+}.
\end{aligned}$$

It follows from these formulas (especially the first) that the space

$$W(\tau) = \operatorname{Span}\{dU_\tau\,(u) \cdot v \mid v \in V_\tau,\ u \in \mathcal{U}(\mathfrak{g}^{\mathbb{C}})\} \tag{1.1.4}$$

is contained in any invariant subspace. In particular, $W(\tau)$ is irreducible.

Let V_τ and V_{τ_1} be finite-dimensional (irreducible) $\mathcal{U}(\mathfrak{k}^{\mathbb{C}})$-modules, and let D be a constant coefficient holomorphic differential operator on $\mathfrak{p}^-$ with values in $\operatorname{Hom}(V_\tau, V_{\tau_1})$.

Definition 1.1.5. *D: $\mathcal{P}(V_\tau) \to \mathcal{P}V_{\tau_1})$ is covariant iff*
$$\forall x \in \mathfrak{g}^{\mathbb{C}}: D\,dU_\tau(x) = dU_{\tau_1}(x)D.$$

Along with $\mathcal{P}(V_\tau)$ we consider the space $\mathcal{E}(V_{\tau'})$ of holomorphic constant coefficient differential operators on $\mathfrak{p}^-$ with values in the contragredient module, $V_\tau' = V_{\tau'}$, to V_τ. For $p \in \mathcal{P}(V_\tau)$ and $q \in \mathcal{E}(V_{\tau'})$ let

$$(q,p) = (q(\frac{\partial}{\partial z}),p(\cdot))\ (0). \tag{1.1.6}$$

This bilinear pairing clearly places $\mathscr{P}(V_\tau)$ and $\mathscr{E}(V_{\tau'})$ in duality and as a result, $\mathscr{E}(V_{\tau'})$ becomes a $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})$-module. The following result is straightforward. See [H&J].

Proposition 1.1.7. *As $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})$-modules,*

$$\mathscr{P}(V_\tau)' = \mathscr{E}(V_{\tau'}) = M(V_{\tau'}).$$

The following is proved in [JV].

Proposition 1.1.8. *A homomorphism φ: $M(V_{\tau_1'}) \to M(V_{\tau'})$ gives rise, by duality, to a covariant differential operator D_φ: $\mathscr{P}(V_\tau) \to \mathscr{P}(V_{\tau_1})$, and conversely.*

Through the results of Bernstein, Gelfand, and Gelfand [B,G,&G] this proposition yields a condition (»condition (A)«) which must be satisfied in order that there may be a covariant differential operator. This observation was crucial in the determination of the full set of unitarizable highest weight modules [JIV] (see also [JII] and [JIII]):

For Λ_0 fixed it is known through the results of Harish-Chandra [H-C] that the modules $W(\tau) = W(\tau)\,(\lambda)$ are unitarizable for λ sufficiently negative. Due to the polynomial behavior, as a function of λ, of the restriction of the hermitian form to finite-dimensional subspaces of $\mathscr{P}(V_\tau)$ (where V_τ, as a vector space, is independent of λ), it follows that the first $\lambda = \lambda_1$ where the hermitian form becomes degenerate (»the first possible place of non-unitarity« – though of course a place at which there is unitarity) is a place where $W(\tau)\,(\lambda_1) \neq M(V_\tau)$. It follows that the annihilator $W^0\,(\tau)$ of $W(\tau)$ is non-trivial. Hence, there is a covariant differential operator or, equivalently, »condition (A)« must be satisfied. The point λ_1 is then easily determined through a diagrammatic presentation of Δ_n^+; described in [JIV]. Furthermore, by looking at the first $\lambda = \lambda_0$ where there is a first order covariant differential operator (»the last possible place of unitarity«; [JIII]) and by paying attention to the exact forms of the homomorphisms at λ_0 and λ_1, one may in fact determine the full set of points above λ_1 at which there is unitarity. In particular, λ_0 is a such. The complete proof also relies on the results in [K&V], [R&V], and [W]. (A different proof has been given in [E, H, & W]).

Let I be an invariant subspace; $W(\tau) \subset I \subset M(\tau)$, and assume that all inclusions are proper. The annihilator I^0 is then non-trivial, and it makes sense to talk about the lowest order elements in I^0. These elements must be annihilated by $\mathfrak{p}^+$ and it follows that there is at least one homomorphism into $\mathscr{P}(V_\tau)'$ whose image is contained in I^0. In case I^0 is a union of such images of homomorphisms, the description by covariant differential operator is then complete. Furthermore, the $\mathfrak{k}$-types of I can be determined as those that are dual to »the $\mathfrak{k}$-types not contained in I^0«, and the $\mathfrak{k}$-types of I^0 can be computed essentially just from the knowledge of the $\mathfrak{k}$-types of modules of the form $\mathscr{P}(V_{\tilde{\tau}})$. One must examine, though, exactly how the images overlap.

However, it may happen that I^0 is not completely covered by the images of homomorphisms. An example of this phenomenon is given in [B&C], and there an example of a reducible socle is also furnished. Some other peculiarities are exemplified in [JV]. All the same, it is clear that it is of importance to know as much about covariant differential operators as possible.

1.2. Basic covariant differential operators

By a scalar module we mean a module $M(V_\tau)$ for which $\dim V_\tau = 1$ or, equivalently; $\tau = \tau(0,\lambda)$. In this section we quote the results of [JV] concerning the set of homomorphisms originating from, or terminating in, scalar modules.

Let $\gamma_1 = \beta, \gamma_2,\dots,\gamma_r$ be a maximal set of orthogonal roots in Δ_n^+, constructed so that γ_i is the element in $\Delta_n^+ \cap \{\gamma_1,\dots,\gamma_{i-1}\}^\perp$ with the smallest height; $i = 2,\dots,r$. Let $\delta_i = \gamma_1 + \dots + \gamma_i$; $i = 1,\dots,r$.

Proposition 1.2.1. ([Smd]). *The set of highest weights of the irreducible submodules of the $\mathfrak{k}^{\mathbb{C}}$-module $\mathscr{U}(\mathfrak{p}^-)$ are*

$$\{-i_1\delta_1-\cdots-i_r\delta_r \mid (i_1,\dots,i_r) \in (\mathbb{Z}_+)^r\}.$$

There are no multiplicities.

Let p denote the dimension of an »off-diagonal« root space in $\mathfrak{g}$ for a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ (cf. [R&V; (2.2.2)]), and let $\lambda_s = -(s-1)\cdot p/2$; $s=1,\dots,r$.

Proposition 1.2.2.

a) *If there is a non-trivial homomorphism*

$$M\big(V_{(0,\lambda)-\Sigma_{s=1}^r i_s\delta_s}\big) \to M(V_{(0,\lambda)})$$

then at most one i_s is different from 0.

b) *There is a non-zero homomorphism*

$$M(V_{(0,\lambda)-n\delta_s}) \to M(V_{(0,\lambda)})$$

exactly when $\lambda = \lambda_s + (n-1)$ *where* λ_s *is given as above and* $n \in \mathbb{N}$.

Let ω_1 be the Weyl group element that satisfies

$$\omega_1(\beta) = \gamma_r \ ; \ \omega_1(\Delta_c^+) = \Delta_c^-. \tag{1.2.3}$$

Let

$$\hat{\gamma}_i = \omega_1(\gamma_i) \ ; \qquad i=1,\dots,r. \tag{1.2.4}$$

Proposition 1.2.5. *There exists a non-zero homomorphism* $M(V_{(0,\lambda)}) \to M(V_\tau)$ *exactly when* $\tau \equiv (0,\lambda) + n\omega_1(\delta_i)$ *for some* $n \in \mathbb{N}$, $i \in \{1,\dots r\}$ *and* $\lambda = \lambda_i - n - \varrho(\hat{\gamma}_i)$. *The homomorphism is unique.*

1.3. The general case on $\mathcal{D}$; an approach

Whereas the Jantzen-Zuckerman translation functor itself, when applied to the results of the previous chapter, does not quite yield the detailed information that we are seeking about general $M(V_\tau)$'s, it is still natural to apply the idea of tensoring with finite-dimensional representations along the lines of ([V; Lemma 4.5.9]) to the present situation. As we shall see, it is in fact possible, by applying such ideas to the dual modules, to obtain a tool which, in particular for some of the classical groups, is remarkably powerful.

Let $\tau = \tau_{(\Lambda_0,\lambda)}$ be fixed, let λ_3 be determined by $(\Lambda_0',\lambda_3)(H_\beta) = 0$, and choose $\lambda_4 \geq \lambda_3$ such that $\lambda_4 - \lambda_3$ is an integer. Then (Λ_0',λ_4) is the highest weight of an irreducible finite-dimensional representation $F(\Lambda_0',\lambda_4)$ of $\mathfrak{g}^{\mathbb{C}}$. Observe the following simple facts:

Lemma 1.3.1. *a) For the dual module $F'(\lambda_0',\lambda_4)$ we have*

$$F'(\Lambda_0',\lambda_4) \simeq \mathbb{W}(\tau_1), \tag{1.3.2}$$

where $\tau_1 = \tau_{(\Lambda_0,\lambda_3-\lambda_4)}$ and $\mathbb{W}(\tau_1)$ *is given by* (1.1.4).
b) The $\widetilde{K}$-type $\tau_2 = \tau_{\Lambda_2}$ which is annihilated by $\mathfrak{p}^+$ in $\mathbb{W}(\tau_1)$ satisfies

$$\Lambda_2 = (\tilde{\Lambda_0}',\lambda_4) = -\omega(\Lambda_0',\lambda_4) \tag{1.3.3}$$

where ω is the Weyl group element which maps the negative Weyl chamber onto the positive, and $\tilde{\Lambda_0}$ is a dominant integral weight of $\mathfrak{k}_1$.
c) τ' is of highest weight (Λ_0',λ') with

$$\lambda' = -(\Lambda_0,\lambda)(H_\beta). \tag{1.3.4}$$

Proof. a) It is obvious that $F(\Lambda_0',\lambda_4)$ is the irreducible quotient of $M(V_{(\Lambda_0',\lambda_4)})$ and hence it follows by Proposition 1.1.7 and (1.1.4) that (1.3.2) holds with a τ_1 of the form $\tau_1 = \tau_{(\Lambda_0,\tilde{\lambda})}$ for some $\tilde{\lambda}$. Let ω_1 be the

Weyl group element that satisfies (1.2.3). Then clearly $(\Lambda_0,\tilde{\lambda}) = -\omega_1(\Lambda_0',\lambda_4)$. Hence $\tilde{\lambda} = (\Lambda_0,\tilde{\lambda})\ (H_{\gamma_r}) = -(\Lambda_0',\lambda_4)\ (H_\beta) = -(\lambda_4-\lambda_3)$.
b) It is obvious that $\Lambda_2 = -\omega(\Lambda_0',\lambda_4)$, and this equals $(\tilde{\Lambda_0}',\check{\lambda})$ for some $\check{\lambda}$. What needs to be proved is that $\check{\lambda} = \lambda_4$. This, however, follows by observing that $\omega(\gamma_r) = -\gamma_r$ since γ_r is the highest root.
c) This follows as in a). Q.E.D.

As $\mathscr{U}(\mathfrak{k}_1{}^{\mathbb{C}})$-modules, $V_{(\Lambda_0,\lambda)} = V_{(\Lambda_0,0)}$. Hence, by Lemma 1.3.1, the elements of $W(\tau_1)$ may be considered as taking values in $V_{\tau'}'$. We denote the duality by $(.,.)$ and observe that we thereby may associate, to any $p \in \mathscr{P}(V_{\tau'})$ and $q \in W(\tau_1)$, a $\mathbb{C}$-valued polynomial

$$(p,q)\ (z) = (p(z),q(z))\ \text{(pointwise)} \tag{1.3.5}$$

Proposition 1.3.6. *Let* $p \in \mathscr{P}(V_{\tau'})$, $q \in W(\tau_1)$, *and let* $x \in \mathfrak{g}^{\mathbb{C}}$. *Then*

$$(dU_{\tau'}(x)p,q) + (p,dU_{\tau_1}(x)q) = dU_{(0,-\lambda-\lambda_4+\lambda_3)}\ (x)(p,q).$$

Proof. Let h_0 denote the element of the center of $\mathfrak{k}$ as in section 1.1. Since ω_1 preserves the set of positive non-compact roots, $\omega_1(h_0) = h_0$. Equation (1.3.7) now follows from (1.1.3) together with the observation that $d\tau'(h_0)$ on $V_{\tau'}$ is given by $-\sqrt{-1}\ \omega_1\ (\Lambda_0,\lambda)\ (h_0) = -\sqrt{-1}(\Lambda_0,\lambda)(h_0)$, whereas $d\tau_1\ (h_0)$ on V_{τ_1} is given by $\sqrt{-1}\ (\Lambda_0,\lambda_3-\lambda_4)\ (h_0)$. Q.E.D.

Let V_{τ_2} denote the $\tilde{K}$-type in $W(\tau_1)$ which is annihilated by $\mathfrak{p}^+$.

Corollary 1.3.8. *Let* $p \in W(\tau')$ *and let* $q \in V_{\tau_2}$. *Then*

$$(p,q) = \sum_{i=1}^{n} d\ U_{(0,-\lambda-\lambda_4+\lambda_3)}(u_i)\ (v_i,q_i)$$

for some elements $u_1,\dots,u_n \in \mathscr{U}(\mathfrak{p}^+)$, $v_1,\dots,v_n \in v_{\tau'}$, *and* $q_1,\dots,q_n \in V_{\tau_2}$.

Proof. By (1.1.4), it suffices to take p of the form $dU_{\tau'}(u)\cdot v$ for $u \in \mathscr{U}(\mathfrak{g}^{\mathbb{C}})$ and $v \in V_{\tau'}$. Since $\mathscr{U}(\mathfrak{g}^{\mathbb{C}}) = \mathscr{U}(\mathfrak{p}^+)\mathscr{U}(\mathfrak{p}^-)\mathscr{U}(\mathfrak{k}^{\mathbb{C}})$ by Poincaré-Birkhoff-Witt, we may assume that $u \in \mathscr{U}(\mathfrak{p}^+)$. Thus, the statement follows directly from Proposition 1.3.6. Q.E.D.

To apply this, observe that by Proposition 1.2.2 and Proposition 1.1.7 there are certain values of λ_0 at which there is a finite number of invariant subspaces $I^i_{\lambda_0}$; $i = 1,\dots$, of $\mathscr{P}(V_{(0,\lambda_0)})$. These are given as the kernels of covariant differential operators. Moreover, since there are no multiplicities at this level, for each $I^i_{\lambda_0}$ there is an n and a δ_{i_0} such that, in the terminology of Proposition 1.2.1, the $\mathfrak{k}$-types of $I^i_{\lambda_0}$ exactly are those whose contragredient representation does not contain $-n\delta_{i_0}$. That is, if the contragredient $\mathfrak{k}$-type is of the form $(0,-\lambda_0) - \Sigma_{i=1}^r n_i\delta_i$, then $\Sigma_{i=i_0}^r n_i <$ n. Equivalently, the $\mathfrak{k}$-types are exactly those which are not in the ideal generated by the polynomials in the $\mathfrak{k}$-type generated by $p^n_{-\delta_s}$. (By choosing the γ_i's of Proposition 1.2.1 differently we could of course avoid having to go to the dual picture.)

It follows by Proposition 1.3.6 that for each $I^i_{\lambda_0}$,

$$\{p \in \mathscr{P}(V_{\tau'}) \mid \forall q \in W(\tau_1): (p,q) \in I^i_{\lambda_0}\} \tag{1.3.9}$$

is an invariant subspace ($\lambda_0 = -\lambda-\lambda_4+\lambda_3$). Further, if the space

$$\mathscr{S}_{\lambda_0} = \mathrm{Span}\{\ (v,q(z)) \mid v \in V_{\tau'},\ q \in V_{\tau_2}\} \tag{1.3.10}$$

(still, $\lambda_0 = -\lambda-\lambda_4+\lambda_3$) is contained in $I^i_{\lambda_0}$ for some i, then by Corollary 1.3.8.

$$\{(p(z),q(z)) \mid p \in W(\tau')\ ,\ q \in V_{\tau_2}\} \tag{1.3.11}$$

is also contained in $I^i_{\lambda_0}$. Since $I^i_{\lambda_0}$ as a set of $\mathfrak{k}$-types is equal to those that do not occur in a certain ideal, it is clear that *$W(\tau')$ cannot equal the full set $\mathscr{P}(V_{\tau'})$* since we can choose $p_0 \in \mathscr{P}(V_{\tau'})$ with coordinate functions in the mentioned ideal, and then $(p_0(z),q(z)) \notin I^i_{\lambda_0}$.

We shall give an example of how to use the last observation. First observe that by Lemma 1.3.1, for $k \in \widetilde{K}$,

$$\begin{aligned}&(U_{(0,-\lambda-\lambda_4+\lambda_3)}\ (k)\ (v,q(\cdot)))\ (z)\\ &= (\tau'(k)v,\tau_1(k)q(k^{-1}z)) = (\tau'(k)v,(\tau_2(k)q)\ (z)),\end{aligned} \tag{1.3.12}$$

i.e. the $\widetilde{K}$-types of $\mathscr{S}_{\lambda_0}$ are contained in $\tau' \otimes \tau_2$.

We now specialize to $Sp(n,\mathbb{R})$. Assertions about V_{τ_2} analogous to the one below can also be made for SU(p,q) and for »most« of the finite-dimensional representations of $SO^\star(2n)$. Also observe that the following remark in fact itself deals with a significant subset of the modules that are the target of this chapter:

Let $\mathfrak{g} = sp(n,\mathbb{R})$. Based on the imbedding of $Sp(n,\mathbb{R})$ into SU(n,n) we choose the following conventional realization of $\mathfrak{g}$ according to which

$$\begin{aligned}\mathfrak{k} &= \{\left(\begin{smallmatrix} ih & 0\\ 0 & -i{}^th\end{smallmatrix}\right) \mid h = h^\star \in M(n,\mathbb{C})\},\\ \mathfrak{p}^- &= \{\left(\begin{smallmatrix} 0 & 0\\ z & 0\end{smallmatrix}\right) \mid z = {}^tz \in M(n,\mathbb{C})\},\ \text{and}\\ \mathfrak{p}^+ &= \{\left(\begin{smallmatrix} 0 & w\\ 0 & 0\end{smallmatrix}\right) \mid w = {}^tw \in M(n,\mathbb{C})\}.\end{aligned}$$

Let τ_1 and τ_2 be as in Lemma 1.3.1.

Lemma 1.3.13. *For* $v \in V_{\tau_1}$ *the prescription*

$$\varphi_v\colon z \to \tau_1(z^{-1})v$$

defines a V_{τ_1}*-valued polynomial which belongs to* V_{τ_2}.

Proof. Observe that in the present situation, $\tau_1' = \tau_2$. Since τ_2 corresponds to the highest weight of a finite dimensional $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})$-module, it is clear that it is polynomial. For $u \in U(n)$ we have

$$\tau_1(u)\tau_1(({}^tuzu)^{-1})v = \tau_1(z^{-1})\tau_1({}^tu^{-1})v,$$

hence it is clear that φ_v transforms according to τ_2. Finally observe that for $x = \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{p}^+$,

$$[x,z] = \begin{pmatrix} xz & 0 \\ 0 & zx \end{pmatrix}, \text{ and } [[x,z],z] = \begin{bmatrix} 0 & 0 \\ -2zxz & 0 \end{bmatrix}.$$

Hence, by (1.1.3)

$$(dU_{\tau_1}(x)\varphi_v)(z) = d\tau_1(xz)\tau_1(z^{-1})v - \tfrac{1}{2}\, d\tau_1(2xz)\tau_1(z^{-1})v = 0. \text{ Q.E.D.}$$

Let $e_1,\ldots,e_n$ denote the standard orthonormal basis of $\mathbb{R}^n$.
Then

$$\Delta_c^+ = \{e_i - e_j \mid 1 \leq i < j \leq n\}, \text{ and}$$
$$\Delta_n^+ = \{e_i + e_j \mid 1 \leq i \leq j \leq n\}.$$

$\Lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n)$ is $\mathfrak{k}_1$-dominant and integral if and only if $\lambda_1 \geq \ldots \geq \lambda_n$ and $\lambda_i - \lambda_j \in \mathbb{Z}$. $\varrho = (n,n-1,\ldots,1)$, and $\lambda = \lambda_1$.

Example. Consider $Sp(4,\mathbb{R})$ and let $\Lambda = (\lambda,\lambda,\lambda-1,\lambda-2)$. We put $\lambda_3 = \lambda_4$ and observe that

$$\tau_1 = (0,0,-1,-2), \text{ and } \tau_2 = (2,1,0,0).$$

Tensor products are computed by means of the Littlewood-Richardson rule (see e.g. [Jms]) which also gives the full solution to $\tau_a \otimes (?) = \tau_b$. To begin with, then, we observe that according to (1.3.12) and Proposition 1.2.1, the possible $\mathfrak{k}$-types of $\mathscr{S}_{-\lambda}$ are $(-\lambda,-\lambda,-\lambda,-\lambda) \otimes (4,2,0,0)$ and $(-\lambda,-\lambda,-\lambda,-\lambda) \otimes (3,3,0,0)$. Since, if e denotes the highest weight vector for τ_2, $(e,\tau_1(z^{-1})e) = (e,\tau_2(z)e)$ is a highest weight vector as a $\mathbb{C}$-valued polynomial, it follows that the first of these does occur. Further, it is easy to see that the corresponding set of polynomials does not exhaust $\mathscr{S}_{-\lambda}$. Hence both of these types occur.

Now, the smallest λ at which a sequence of elements from Δ_n^+ can satisfy condition (A) for any pair $(\cdot,\Lambda+\varrho)$ is evidently $\lambda = -3$, corresponding to $\{e_3+e_4\}$. However, this does not correspond to a highest weight of $\mathfrak{p}^- \otimes V_\tau$ (cf. the proof of Proposition 7.3 in [JIII]). It is also straightforward to see that at $\lambda = -5/2$ there can be no sequence satisfying condition (A) for a pair $(\Lambda_1+\varrho,\Lambda+\varrho)$ with Λ_1 $\mathfrak{k}_1$-dominant. We will in this example study the values $\lambda = -2,-3/2,-1,-1/2$, and 0. Note that the value of p in Proposition 2.2 is 1 for $sp(n,\mathbb{R})$.

$\lambda = -2$: Consider the imbedding of $sp(4,\mathbb{R})$ into $sp(5,\mathbb{R})$ for which, if $e_1,\ldots,e_5$ is a basis of $\mathbb{R}^5$ as above, the space $\mathfrak{p}^+$ for $sp(4,\mathbb{R})$ is contained in the $\mathfrak{p}^+$ for $sp(5,\mathbb{R})$, and corresponds to the roots e_i+e_j with $2 \le i,j$. Let $\widetilde{\Lambda} = (\lambda,\lambda,\lambda,\lambda-1,\lambda-2)$ $(\lambda = -2)$ and proceed with $sp(5,\mathbb{R})$: It follows from Proposition 1.2.2 and the remarks following (1.3.11) that $W(\tau')$ cannot equal the full set $\mathscr{P}(V_{\tau'})$. The invariant subspace $I^i_{\lambda_0}$ to be used for this argument is, as a space of polynomials, a complement to the ideal generated by the polynomial representation contragredient to $(-2,-2,-2,-2,-2)$. ($(-2,-2,-2,-2,-2)$ is, in the language of Proposition 1.2.1, equal to $-\delta_5$ and corresponds to the one-dimensional representation space $\mathbb{C}\cdot\det z$. $-\delta_4 = (0,-2,-2,-2,-2)$ corresponds to 4×4 minors of z, etc.) We have that $(2,1,0,0,0) \otimes (2,2,2,1,0)$ contains $(2,2,2,2,2)$ and it is, anyway, straightforward to see that the only possible choice of a sequence of elements of Δ_n^+ satisfying condition (A) for a pair $(\Lambda_1+\varrho,\widetilde{\Lambda}+\varrho)$ has $\Lambda_1 = (-2,-2,-2,-2,-2) \otimes (0,-1,-2,-2,-2)$. Finally, since the highest weight vector in $\mathfrak{p}^- \otimes \mathfrak{p}^- \otimes V_\tau$ corresponding to this Λ_1 actually only lives on the $sp(4,\mathbb{R})$ above, it follows that we do have a non-zero homomorphism

$$M(V_{(-3,-4,-4,-4)}) \to M(V_\tau),$$

and there can only be one such since Λ_1 has multiplicity one in $\mathscr{U}(\mathfrak{p}^-) \otimes V_\tau$.

$\lambda = -3/2$: This may be treated analogously by using the ideal generated by the polynomial representation contragredient to $(0,-2,-2,-2,-2)$ (4×4 minors). But there is no need to pass to $sp(5,\mathbb{R})$; $(-2,-2,-2,-2)$ for $sp(4,\mathbb{R})$ can also be used (then it is just the determinant). However, this is also a point at which a first order differential operator exists by

Proposition 1.6 in [JV]. Specifically, $\lambda = e_2 + e_3$. Moreover, it is easy to see that the corresponding homomorphism is the only possible.

$\lambda = -1$: It follows from Proposition 1.5 in [JV] (or [B,G,&G]), by trial and error, that there can be no homomorphisms for this value.

$\lambda = -1/2$: By looking at the ideal generated by the polynomials in the $\mathfrak{k}$-type whose contragredient is $(-4,-4,-4,-4)$ $(= -2\delta_4)$ it follows that $W(\tau') \neq \mathcal{P}(V_{\tau'})$. Moreover, one can see that the $\mathfrak{k}$-type $\Lambda_1 = (1/2,1/2,1/2,1/2) \otimes (4,4,3,2)$ does not belong to $W(\tau')$ and one can find a sequence of elements of Δ_n^+ satisfying condition (A) for the pair $(\Lambda_1'+\varrho,\Lambda+\varrho)$. However, there are other sequences also satisfying this condition but corresponding to lower order differential operators. In fact, there is a first order differential operator corresponding to $\lambda = e_1+e_3$. This phenomenon seems to be quite typical: When the method fails to give more information than $W(\tau') \neq \mathcal{P}(V_{\tau'})$ (which just implies the existence of *some* covariant differential operator) one can usually quite easily establish the existence of the lowest order operator and, more generally, obtain a sequence of differential operators of lowest possible degree. The present situation furnishes an example of this: There is a system of non-zero homomorphisms

$$M(V_{(-1/2-3,-1/2-4,-1/2-4,-1/2-4)}) \xrightarrow{\varphi_1} M(V_{(-1/2-2,-1/2-3,-1/2-4,-1/2-4)}) \xrightarrow{\varphi_2}$$
$$M(V_{(-1/2-1,-1/2-3,-1/2-3,-1/2-4)}) \xrightarrow{\varphi_3} M(V_{(-1/2-1,-1/2-1,-1/2-1,-1/2-4)}) \xrightarrow{\varphi_4}$$
$$M(V_{(-1/2,-1/2-1,-1/2-1,-1/2-3)}) \xrightarrow{\varphi_5} M(V_{(-1/2,-1/2,-1/2-1,-1/2-2)}).$$

Of these, all but φ_3 correspond to first order. The existence of φ_3 follows along the lines of the cases $\lambda = -3/2$ and $\lambda = -2$. Finally, by looking at the images of the various homomorphisms and observing that everywhere there is multiplicity one, it is easy to conclude that $\varphi_5 \circ \varphi_4$, $\varphi_5 \circ \varphi_4 \circ \varphi_3$, $\varphi_5 \circ \varphi_4 \circ \varphi_3 \circ \varphi_2$, and $\varphi_5 \circ \varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1$ all are non-zero and that they, along with φ_5, constitute the full set of homomorphisms into $M(V_\tau)$.

$\lambda = 0$: There is a sequence of non-zero homomorphisms

$$M(V_{(-3,-4,-5,-5)}) \xrightarrow{\varphi_1} M(V_{(-1,-2,-3,-5)}) \xrightarrow{\varphi_2}$$
$$M(V_{(0,-2,-3,-4)}) \xrightarrow{\varphi_3} M(V_{(0,0,-1,-2)}).$$

The existence of φ_1 and φ_3 follows as above, and φ_2 corresponds to a first order operator. It might seem that there could be two distinct homomorphisms at the level of φ_1, but it follows easily from Proposition

1.5 in [JV] that $M(V_{(-3,-4,-5,-5)})$ is irreducible, and the existence of two distinct homomorphisms would contradict this. By looking at multiplicities it follows that $\varphi_3 \circ \varphi_2$ is non-zero and, moreover, is the unique such. However, $\varphi_3 \circ \varphi_2 \circ \varphi_1$ is zero. To wit, $V_{(-3,-4,-5,-5)}$ does not occur in $\mathcal{U}(\mathfrak{p}^-) \otimes V_{(0,0,-1,-2)}$, and thus there can be no non-zero homomorphism at this level.

1.4. Conformal covariance

Given a generalized highest weight module $M(V_\tau)$, Bernstein, Gelfand, and Gelfand [B,G,&G] gives the highest weights of the $\mathfrak{k}$-types that may possibly be annihilated by $\mathfrak{p}^+$ and thus define homomorphisms into $M(V_\tau)$. As described in the previous section, the results of [JV] are quite useful in the description of the set of homomorphisms; indeed, in many cases it yields directly the full set. However, there are some complicated situations, e.g. those in which one (or several) of the »BGG $\mathfrak{k}$-types« occurs with multiplicity greater than one. We believe that there is a principle which can handle those situations, and whose applicability goes even further. It should be noted that there are no examples of multiplicity greater than one in the sets of homomorphisms as above. The principle in its mildest formulation states (tube domain case) that only for very special Λ_0's can it happen that a $\mathfrak{k}$-type $p \in M(V_\tau)$ of the form $p = u \cdot q$ with $q \in M(V_\tau)$ and $u \in \mathcal{U}(\mathfrak{p}^-)^{\mathfrak{k}_1}$ is annihilated by $\mathfrak{p}^+$. Actually, this principle was the main motivation behind the results in [JV].

We will now furnish an example based on the Lie algebra su(2,2) of the conformal group. Here the formulation is quite precise and may in fact be proved to be sufficient to determine the full set of conformally covariant differential operators. Let

$$\square = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} .$$

Consider those covariants that operate on spin $(\frac{n}{2}, \pm)$:

Proposition 1.4.1. *The only covariant differential operators that contain $\square$ (to some power) as a factor are those that intertwine spin $(\frac{n}{2}, \pm)$ with spin $(\frac{n}{2}, \pm)$;* $n=0,1,2,\dots$.

The proof of this proposition will appear elsewhere. The situations corresponding to $n=0$ and $n=1$ are described in [J&V;I].

Let e_1, e_2, and e_3 denote the standard orthonormal basis of $\mathbb{R}^3$. Then

$$\Delta_c^+ = \{(e_2-e_3),(e_2+e_3)\} \text{ , and}$$
$$\Delta_n^+ = \{\beta = (e_1-e_2),\ \alpha_1 = (e_1-e_3),\ \alpha_2 = (e_1+e_3),\ \gamma_r = (e_1+e_2)\}$$

($su(2,2)$ is of type A_3) . $\Lambda = (\lambda_1,\lambda_2,\lambda_3)$ is $\mathfrak{k}_1$-dominant and integral if and only if $\lambda_2 \pm \lambda_3 \in \{0,1,\ldots\}$, and $\varrho = (2,1,0)$. For $\alpha \in \Delta$ we denote the reflexion corresponding to α by S_α. We label, for convenience, the homomorphisms by a sequence of reflexions involved in the corresponding »condition (A)«.

Example. $\Lambda = (\lambda,1,1) \qquad (\lambda \in \mathbb{R})$.

At $\lambda = -3$, S_{γ_r} satisfies »condition (A)« but $S_{\gamma_r}(\Lambda+\varrho)-\varrho$ does not correspond to an element of $\mathfrak{p}^- \otimes V_\tau$.

At $\lambda = -2$, S_{γ_r} and $S_{\gamma_r}S_{\alpha_2}$ are ruled out for the same reason, but S_{α_2} defines a first order covariant differential operator.

At $\lambda = -1$, the situation is similar to $\lambda = -2$; the only difference is that S_{α_2} here gives an operator of order 2.

At $\lambda = 0$, $S_{\alpha_1}S_{\alpha_2}$ corresponds to the situation in Proposition 1.4.1 and is, moreover, the only sequence which corresponds to an element of the module.

Aτ $\lambda \in \mathbb{N}$, S_β, $S_{\alpha_1}S_{\alpha_2}$, and $S_{\alpha_1}S_{\alpha_2}S_\beta$ correspond to elements of the module, However, due to the principle, only S_β and $S_{\alpha_1}S_{\alpha_2}$ survive.

2. Irreducible mixtures of unitarizable modules

Consider the following very general situation: One is given two groups, G_1 and G_2, a family $(\Pi_\alpha, V_\alpha)_{\alpha\in A}$ of representations Π_α of G_1 on spaces V_α, and a representation π_m of G_2 on $V = \underset{\alpha\in A}{V_\alpha}$. Let us insist that A contains more than one element and assume that it makes sense to inquire about irreducibility or indecomposability. One may then talk about V having one of these properties with respect to either G_2 or $G_1\times G_2$, and one may say that G_2 makes V into an irreducible or indecomposable module of G_1-representations, respectively. Naturally, additional assumptions about the representations may be inserted, e.g. unitarity.

Let us from now on restrict ourselves to the case in which G_1 is isomorphic to a subgroup $\widetilde{G}_1$ of G_2 and assume that the representations of G_1 are irreducible. Those representations of Chapter 1 that have invariant subspaces defined by covariant differential operators, furnish examples for which $\widetilde{G}_1 = G_2$ and A is finite. In these cases, the representation of G_2 is never unitary. On the other hand one may take a unitary irreducible representation of G_2 and restrict it to G_1, but this, especially when G_1 is non-compact, tends to give sets A which are uncountable.

Below we present, through some simple examples, two different circumstances under which one in a natural way gets irreducible mixtures of unitary representations, and where A is countable. In both examples $\widetilde{G}_1$ may be taken to be equal to G_2 and in the first example, A is in fact *finite*.

2.1. Representations which do not remain irreducible when restricted to the extended Poincaré group.

The general situation is the following: G_2 is (a covering group of) a group of holomorphic transformations of an irreducible hermitian symmetric space $\mathcal{D}$ of the non-compact type. Assume for simplicity that $\mathcal{D}$ is of tube type; $\mathcal{D} = \mathbb{R}^n + iV$ for an open convex cone $V \in \mathbb{R}^n$. Let P be the maximal parabolic subgroup of G_2 which contains the translations $L_{x_0}(x+iv) = x + x_0 + iv$ of $\mathcal{D}$, for all $x_0 \in \mathbb{R}^n$. Consider an irreducible unitary representation π_m of G_2 on a space of vector valued holomorphic functions on $\mathcal{D}$. Then $\pi_m|_P = \bigoplus_{i=1}^{N} \pi_{i,m,P}$ for some $N \in \mathbb{N}$, with $\pi_{i,m,P}$ irreducible and unitary for all $i=1,\ldots,N$. In some special cases, $N=1$, and for some of the most singular of these, the representation remains irreducible when restricted to a normal subgroup P_0 of P for which P/P_0 is isomorphic to the one-dimensional center of the reductive (linear) part of P. (Cf. below.)

The decomposition of the restriction of π_m to P is handled by imbedding π_m into a degenerate principal series representation, and this is accomplished by taking boundary values on the Shilov boundary of $\mathcal{D}$: $\lim_{v\to 0} f(x+iv)$ (which exists at least on a dense set of functions). See [JV;I] for an example. Furthermore, each representation $\pi_{i,m,P}$ is recognized as the restriction of an irreducible unitary holomorphic representation of G_2 to P. We denote the last group by G_1 since is should really be thought of as distinct (in fact, $G_1 \cap G_2 = P$); and thus get the promised phenomenon. One interesting question to which we do not know the answer is: How big is the group generated by G_1 and G_2; does it have a geometric interpretation?

Let us now be specific: In the following formulas, the letters a,b,c,d,x,y,z, and w denote 2×2 complex matrices. Moreover, in the following definitions, 0 is the trivial 2×2 matrix, and 1 is the identity.

$$\begin{aligned} G_2 &= SU(2,2) = \{g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid \det g = 1 \text{ and } g^\star\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)g = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\}. \\ P &= \{g = \left(\begin{smallmatrix} a & ax \\ 0 & a^{\star-1} \end{smallmatrix}\right) \mid \det a \in \mathbb{R} \setminus \{0\}, \text{ and } x = x^\star\}. \\ \mathscr{D} &= \{z \mid (z-z^\star)/2i \text{ is strictly positive definete}\} \\ &= \mathbb{R}^4 + iC^+, \end{aligned} \tag{2.1.1}$$

where $C^+ = \{y | y=y^\star, \text{ tr } y > 0 \text{ , and } \det y > 0\}$.

P is isomorphic to a 2-fold covering of the extended Poincaré group. The title of this section is motivated by [M&T], this investigation being complementary to that.

On the space of holomorphic functions on $\mathscr{D}$ with values in $\mathbb{C}^2 \times \mathbb{C}^2$ we consider the one-parameter family of representations U_j of G_2 given by

$$(U_j(g)f)\ (z) = \det\ (cz+d)^{-j}(cz+d)^{-1} \otimes (zc^\star+d^\star)f(g^{-1}z) \tag{2.1.2}$$

for $g^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SU(2,2)$, $g^{-1}z = (az+b)\ (cz+d)^{-1}$, and $j \in \mathbb{Z}$. It follows from [JI, p. 324] that there exists a $K > 0$, independent of j, such that the reproducing kernel $K_j(z,w)$, which is given by

$$K_j(z,w) = \det\ (z-w^\star)/2i)^{-j}\ ((z-w^\star)/i)^{-1} \otimes ((z\text{-}w^\star)/i), \tag{2.1.3}$$

may be written, for $j \geq 3$, as

$$K_j(z,w) = K \cdot \int_{C^+} e^{itr(zw)} F_j(y) dy \tag{2.1.4}$$

with $F_j(y) = \det\ y^{j-3}[y \otimes \tilde{y} + (j-2)^{-1} \det y \cdot T]$; $\tilde{y}$ being the matrix for which $y \cdot \tilde{y} = \det y$ and T being the matrix which, in a basis f_1, f_2, f_3, f_4 of $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$ satisfying that f_1, f_2, f_3 corresponds to the symmetric subspace and f_4 to the antisymmetric, is the diagonal matrix $T = d(1,1,1,-1)$. T then satisfies that $T(a \otimes b) = (b \otimes a)T$ for all a and b. For details about reproducing kernels we refer to [J&V;I]. It is the positive – definiteness of F_j for $j \geq 3$ (for $j = 3$ only semi-definiteness) that implies the unitarity of the representations U_j for $j \geq 3$. Let us from now on assume that $j \geq 4$. The Hilbert space then consists of functions of the form

$$F_f(z) = \int_{C^+} e^{itrzy} f(y) dy \tag{2.1.5}$$

and the inner product is given by

$$<F_{f_1}, F_{f_2}> = \int_{C^+} <F_j^{-1}\ (y) f_1(y), f_2(y) > dy. \tag{2.1.6}$$

The subgroup of SU(2,2) whose elements are of the form

$g = \begin{pmatrix} a & 0 \\ 0 & a^{\star-1} \end{pmatrix}$ (the linear part of P) acts as

$$(U_j(g)F_f)\ (z) = \int_{C^+} e^{itrzy}\ (a^{\star-1}\otimes a)\det a^{4-j}\ f(a^\star ya)dy. \qquad (2.1.7)$$

To make the decomposition under P straightforward, we would like the action on functions on C^+ to be $\det a^{2-j}\ (a\otimes a)f(a^\star ya)$, which is unitary in the inner product

$$<f_1,f_2> = \int_{C^+} <G_j(y)f_1(y),f_2(y)> dy, \qquad (2.1.8)$$

with $G_j(y) = (y\otimes y)\ (\det y)^{-j}$. Thus, we seek an intertwining operator of the form: multiplication by a matrix $M_j(y)$ which satisfies that $M_j(a^\star ya) = (a\otimes a)^{-1}M_j(y)\ (a^{\star-1}\otimes a)$ for all a and all $y \in C^+$. This property is satisfied by any $M_\beta\ (y) = (1+\beta T)\ (\widetilde{y}\otimes 1)$, and it is sufficient to consider this family. To wit, the additional requirement of unitarity; that $M_\beta(y) \cdot G_j(y) \cdot M_\beta(y) = c\cdot F_j^{-1}(y)$ for some $c > 0$, is satisfied provided that $\beta^2+\beta(j-2)+1 = 0$ and $(\beta^2+1)\ (j-2)+\beta > 0$, and this has a solution when $j \geq 4$.

We observe that the representations of G_1 which we obtain are

$$(U_{1,j}(g)f)\ (z) = \det(cz+d)^{-j-2}(zc^\star+d^\star)\otimes_s(zc^\star+d^\star)f(g^{-1}z) \qquad (2.1.9)$$

and

$$(U_{2,j}(g)f)\ (z) = \det(cz+d)^{-j-1}f(g^{-1}z), \qquad (2.1.10)$$

for $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and functions with values in $\mathbb{C}^2 \otimes_s \mathbb{C}^2$ and $\mathbb{C}^2 \otimes_a \mathbb{C}^2$, respectively.

Transformed back to the space of $\mathbb{C}^2 \otimes \mathbb{C}^2$ -valued holomorphic functions on $\mathscr{D}$ the intertwining operator is

$$(1+\beta T)\ (c({}^t D)\ \otimes\ 1), \qquad (2.1.11)$$

$c({}^tD)$ being »one-half« of the Dirac operator as in [J I]. Since any function $f: \mathscr{D} \to \mathbb{C}^4$ can be written as a sum of functions of the form $f_i \otimes v_i$, with f_i: $\mathscr{D} \to \mathbb{C}^2$ and $v_i \in \mathbb{C}^2$ $(i \leq 2)$, we may apply the covariance property of $c({}^tD)$ ([JI]):

Let dV_0 denote the representation of su(2,2) corresponding to the action $(g\cdot f)\ (z) = f(g^{-1}z)$, let dU_j be the representation corresponding to U_j, and consider e.g. $\widetilde{x} = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}$ in su(2,2) $(x = x^\star)$. Then

$$dU_j(\widetilde{x}) = j\ \mathrm{tr}\ xz\ +\ xz\otimes 1\ -\ 1\otimes zx\ +\ dV_0(\widetilde{x}) \qquad (2.1.12)$$

and

$$\begin{aligned}&(c({}^tD)\otimes 1)\ (dU_j(\widetilde{x})) = &(2.1.13)\\&(c({}^tD)\otimes 1)\ (\mathrm{tr}\ xz + xz\otimes 1 + dV_0\ (\widetilde{x}) + (j-1)\ \mathrm{tr}\ xz - 1\otimes zx) =\\&((j+2)\ \mathrm{tr}\ xz - zx\otimes 1 - 1\otimes zx + dV_0\ (\widetilde{x}))\ (c({}^tD)\otimes 1) +\\&(c({}^tD)\otimes 1)\ [\ (j-1)\ \mathrm{tr}\ xz - 1\otimes zx]\ ,\end{aligned}$$

where the last term means $c({}^tD)\otimes 1$ acting on [...]. We recognize here $((j+2)\ \mathrm{tr}\ xz - zx\otimes 1 - 1\otimes zx + dV_0\ (\widetilde{x}))$ as the infinitesimal action corresponding to the representation $(U(g)f)(z) = \det(cz+d)^{-j-2}(zc^\star+d^\star) \otimes (zc^\star+d^\star)f(g^{-1}z)$. To make the computation complete we should of course introduce the inverse to $c({}^tD)\otimes 1$, and this can only be done by returning by the Fourier-Laplace transform to the space of functions on C^+ on which this makes sense. Let us also remark that instead of using a decomposition based on $a \otimes a$ we might as well have used one based on $a^{\star -1} \otimes a^{\star -1}$. Then we would have obtained representations involving $(cz+d)^{-1} \otimes (cz+d)^{-1}$.

We conclude this section by a brief description of the situation when $j = 3$. Let V_4 denote the representation on $\mathbb{C}$-valued functions given by

$$(V_4(g)f)\ (z) = (cz+d)^{-4}f(g^{-1}z). \tag{2.1.14}$$

Then there exists a first order constant coefficient differential operator D such that for all $g \in SU(2,2)$:

$$V_4(g)D = DU_3(g). \tag{2.1.15}$$

U_3 is unitary and irreducible on the kernel of D (inside the space of holomorphic functions) and so is the restriction to P. V_4 is unitary and irreducible on SU(2,2) as well as on P.

We finally mention that there is a non-linear equation left invariant by U_3. Unlike a similar construction for spin ½ given by B. Ørsted and the author, independently, this equation may be taken to be holomorphic:

Let $<.,.>$ be a complex bilinear form on $\mathbb{C}^2 \otimes \mathbb{C}^2$. Consider the 2×2 complex matrix $m = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. Then for all 2×2 complex matrices a, $mam^{-1} = {}^t\widetilde{a}$. For any $c \in \mathbb{C}$ the equation

$$Df = c((m\otimes m)f,f)^{2/3} \tag{2.1.16}$$

is then invariant, as is straightforward to see $(\det(cz+d) = \det(zc^\star+d^\star))$.

2.2. Mixings resulting from compatible imbeddings.

Let $\mathscr{D}$ be a bounded homogeneous domain in $\mathbb{C}^n$ containing the origin 0, let G be a group of holomorphic transformations of G, K the subgroup of G that fixes 0, and assume that $\mathscr{D}$ is homogeneous with respect to G ($\mathscr{D} \approx G/K$). We will say that we have a compatible imbedding of a hermitian symmetric space $\mathscr{D}_1$, (of the non-compact type) into $\mathscr{D}$ if there is a complex submanifold $\mathscr{D}_{1,s}$ of $\mathscr{D}$ containing (for convenience) 0 such that $\mathscr{D}_1$ and $\mathscr{D}_{1,s}$ are bi-holomorphically equivalent and such that $\mathscr{D}_{1,s}$ is homogeneous with respect to the subgroup G_1 of G that leaves $\mathscr{D}_{1,s}$ invariant. We let $K_1 = K \cap G_1$.

Let τ be a unitary representation of K on a finite dimensional vector space V_τ and assume that the representation U_τ of G, obtained from τ through holomorphic induction, is unitary in a Hilbert space H_τ of V_τ-valued holomorphic functions on $\mathscr{D}$. As described in [J&V;II], the decomposition of the restriction of U_τ to G_1 can be obtained from the filtration of H_τ defined by the order of vanishing on $\mathscr{D}_{1,s}$. As a result one gets

$$H_\tau = \bigoplus_{i=1}^{\infty} H_{\tau_i}$$

and

$$U_\tau = \bigoplus_{i=1}^{\infty} U_{\tau_i}$$

where the U_{τ_i}'s are unitary representations of G_1 obtained through holomorphic induction of finite-dimensional unitary representations τ_i of K_1. There may be multiplicities (always finite) and the sum is always at most countable. Evidently, the elements of G outside of G_1 will mix up, through the representation U_τ, the spaces H_{τ_i}. Thus, it is natural to look for another copy of G_1 inside G. This copy does not necessarily have to be of the same nature, i.e. the direct inclusion of SO(4,2) into SU(4,2) does not correspond to a compatible imbedding, but for now we will assume that it is. Even then, the two copies do not have to be conjugate inside G. For instance, if $\mathscr{D} = \mathscr{D}_3 \times \mathscr{D}_3$ then there are at least three interesting and isomorphic submanifolds, namely $\mathscr{D}_3 \times \{0\}$, $\{0\} \times \mathscr{D}_3$, and the diagonal in $\mathscr{D}$, and the corresponding groups are not conjugate. We shall not discuss questions concerning irreducibility here since it is quite clear how such questions should be approached. Rather, we conclude this general discussion with the remark that a K_1-type in a fixed H_{τ_i}, under the action of K, will only travel into a finite number of other H_{τ_i}'s. This is clear from the decomposition.

Let us give a simple example: Let $\{e_1, e_2, e_3, e_4, e_5\}$ be a standard basis of $\mathbb{C}^5$ and let SU(2,3) be the group of linear transformations of $\mathbb{C}^5$ that leaves invariant the sesquilinear form which, in the given basis, is defined by the diagonal matrix d(1,1,-1,-1,-1). We have that $K = \{ (u,v) \in U(2)xU(3) \mid \det u \cdot \det v = 1\}$. As G_1 we take SU(2,2). Copy no. 1 is taken to be the subgroup defined by (e_1, e_2, e_3, e_4) and copy no. 2 to be the one defined by (e_1, e_2, e_4, e_5), though for some purposes it might be more natural to take a more twisted version as copy no. 2. We shall here be content to give the decomposition of some representations of SU(2,3) under copy no. 1. We let z denote a complex matrix with 2 rows and 3 columns; the space $\mathscr{D}$ corresponding to SU(2,3) may be taken to be a bounded subset containing the origin, of the vector space of all such matrices.

Our first example is with $\tau(u,v) = \det u^{-n}$. The corresponding representation is denoted by U_n. Restricted to K it has the form

$$(U_n(u,v)f)\ (z) = \det u^{-n} f(u^{-1}zv). \tag{2.2.1}$$

For $n \in \mathbb{N}$, U_n is unitary. The decomposition is obtained by expanding functions in the variables corresponding to the 3rd column. The set of τ_i's is then equal to $\{\det u^{-n} \overset{j}{\underset{s}{\otimes}} u \mid j=0,1,2,\ldots\}$. This is the case even for $n = 1$, where the representation space is annihilated by a second order differential operator. The reason is that this operator does not contain a summand which is purely a differential operator in the variables corresponding to the 3rd column.

Our second, and final, example is with $\tau = u(\det u)^{-n}$; the corresponding representation U_τ is denoted by V_n. Restricted to K, V_n has the form

$$(V_n(u,v)f)\ (z) = u \det u^{-n} f(u^{-1}zv), \tag{2.2.2}$$

where now f takes values in $\mathbb{C}^2$. For $n \geq 2$, V_n is unitary. For $n > 2$ the set of τ_i's is $\{\det u^{-n} \overset{j}{\underset{s}{\otimes}} u \mid j = 1,2,\ldots\} \cup \{\det u^{-n+1} \overset{j}{\underset{s}{\otimes}} u \mid j=0,1,2,\ldots\}$, but for $n = 2$, where the representation space is annihilated by a first order (matrix valued) differential operator, we only get $\{\det u^{-n} \overset{j}{\underset{s}{\otimes}} u \mid j=1,2,\ldots\}$.

References

B, G, & G: I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, Differential operators on the base affine space and a study of $\mathfrak{g}$-modules, in »Lie Groups and Their Representations« (I. M. Gelfand, Ed.), Adam Hilger; London, 1975.

B&C: B. D. Boe and D. H. Collingwood, A comparison theory for the structure of induced representations, Preprint, 1983.

D&H: P. J. Davis and R. Hersh: *The Mathematical Experience,* Birkhäuser; Boston, 1980.

E,H,&W: T. Enright, R. Howe, and N. Wallach, A classification of unitary highest weight modules, in *Representation Theory of Reductive Groups* (P. C. Trombi, Ed.), Birkhäuser; Boston-Basel-Stuttgart, 1983.

H-C: Harish-Chandra, Representations of semi-simple Lie groups V&VI, Amer. J. Math. **78** (1956), 1-41 and 564-628.

H&J: M. Harris and H. P. Jakobsen, Covariant differential operators, in *Group Theoretical Methods in Physics* (M. Serdaroglu and E. Ínönü, Eds.), Lecture Notes in Physics **180,** Springer; Berlin-Heidelberg-New York-Tokyo, 1983.

J&V;I: H. P. Jakobsen and M. Vergne, Wave and Dirac operators, and representations of the conformal group, J. Funct. Anal. **24** (1977), 52-106.

J&V;II: H. P. Jakobsen and M. Vergne, Restrictions and expansions of holomorphic representations, J. Funct. Anal. **34** (1979), 29-53.

JI: H. P. Jakobsen, Intertwining differential operators for $Mp(n, \mathbb{R})$ and $SU(n,n)$, Trans. Amer. Math. Soc. **246** (1978), 311-337.

JII: H. P. Jakobsen, On singular holomorphic representations, Invent. Math. **62** (1980), 67-78.

JIII: H. P. Jakobsen, The last possible place of unitarity for certain highest weight modules, Math. Ann. **256** (1981), 439-447.

JIV: H. P. Jakobsen, Hermitian symmetric spaces and their unitary highest weight modules, J. Funct. Anal. **52** (1983), 385-412.

JV: H. P. Jakobsen, Basic covariant differential operators on hermitian symmetric spaces, preprint 1984. To appear in Ann. Scient. Éc. Norm. Sup.

Jms: G. D. James, *The Representations Theory of the Symmetric Groups,* Lecture Notes in Math. **682,** Springer; Berlin-Heidelberg-New York, 1978.

K&V: M. Kashiwara and M. Vergne, On the Segal-Shale-Weil representation and harmonic polynomials, Invent. Math. **44** (1978), 1-47.

M&T: G. Mack and I. Todorov, Irreducibility of the ladder representation of $U(2,2)$ when restricted to the Poincaré subgroup, J. Math. Phys. **10** (1969), 2078-2085.

R&V: H. Rossi and M. Vergne, Analytic continuation of the holomorphic discrete series of a semi-simple Lie group, Acta Math. **136** (1976), 1-59.

Smd: W. Schmid, Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen, Invent Math. **9** (1969), 61-80.

S: I. E. Segal, *Mathematical Cosmology and Extragalactic Astronomy,* Academic Press; New York, 1976.

S,J,Ø,P,&S: I. E. Segal, H. P. Jakobsen, B. Ørsted, S. M. Paneitz, and B. Speh, Covariant chronogeometry and extreme distances: Elementary particles, Proc. Natl. Acad. Sci. USA **78** (1981), 5261-5265.

V: D. Vogan, *Representations of Real Reductive Lie Groups,* Birkhäuser; Boston-Basel-Stuttgart, 1981.

W: N. Wallach, Analytic continuation of the discrete series II, Trans. Amer. Math. Soc. **251** (1979), 19-37.